A Method for Construction of Lyapunov Functions for Higher Order Sliding Modes

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***Abstract*— A method to construct (strict) Lyapunov Functions for a class of Higher Order Sliding Modes (HOSM) algorithms, that are homogeneous and piecewise state affine is presented. It is shown first that several HOSM algorithms presented in the literature posses these properties. The basic idea of the construction method is borrowed from the constructive proofs of the Lyapunov's Converse Theorems. It is shown, by means of some concrete examples of second and third order, that the construction of the Lyapunov Function can be done for this class of systems. The obtained Lyapunov functions allow the estimation of the convergence time, the values of the gains that render the origin finite time stable, and the robustness of the algorithms to bounded perturbations.**

Key words: Sliding Mode Control, Lyapunov Functions.

I. Introduction

Lyapunov's direct method is a very general and the most accepted method to determine the stability properties of dynamical systems. It has also become a central tool for analysis and (robust) controller design in the modern control theory.

High Order Sliding Modes (HOSM) provide finite time stability and have an enormous capacity to reject non vanishing disturbances. Moreover, they eliminate the relative degree one restriction of the classical (first order) sliding modes (SM), and they reduce the high frequency switching (Chattering) characteristic of the SM controllers.

Despite of the importance of Lyapunov's method for analysis and design in control the main tools to ascertain the stability and robustness properties of HOSM algorithms are the geometric methods and the homogeneity theory [3], [9], [10]. It is natural to try to develop a Lyapunov-based analysis and design theory for HOSM, and for this to be possible it is crucial to be able to construct Lyapunov functions for HOSM algorithms. Several steps have been done in this direction recently. Weak Lyapunov functions (with negative semidefinite derivative) have been obtained for some second order sliding modes algorithms in [8]. However, weak Lyapunov functions are not completely satisfactory for analysis and design. For the Twisting Algorithm a strong Lyapunov function has been obtained in [15], where basically the classical Zubov's method [1] has been used to construct it. This consists in solving a first order Partial Differential Equation, and then patching the solutions. Solving a PDE is not always an easy task. Some other Lyapunov functions for HOSM algorithms have been proposed in the literature: for the Super-Twisting algorithm in [16], [11], [12]; for the Twisting Algorithm in [15], [17], [13]. However, there is still a big necessity of constructive methods to cover reasonably the area.

The main objective of this paper is to make a further contribution in this field. We propose a construction method of (strict) Lyapunov functions for a class of HOSM algorithms, sharing two properties: being homogeneous (what is today a standard property for HOSM) and being piecewise state affine. This last property is less known in the area, but many important HOSM algorithms have it. For this class of systems we describe a constructive method of Lyapunov functions and we show by means of two examples of order two and three that it effectively provides a usable function. We believe that this method will be able to provide Lyapunov functions for an important family of HOSM algorithms in the near future.

In Section II we illustrate by means of examples that many HOSM algorithms are indeed homogeneous and piecewise state affine. We also describe the construction method and some of its properties. In Section III the method is illustrated by constructing two Lyapunov functions for the Twisting algorithm. A Third order algorithm is treated in Section IV. Finally in Section V we give some conclusions.

II. Motivation and Description of the Construction Method

Due to the strong geometric and dynamical properties of homogeneous systems, most Higher order sliding modes algorithms, that are discontinuous by nature, are designed to be homogeneous [9], [10]. This property, together with a geometrical analysis, has become also the main tool for the stability and robustness analysis of these algorithms.

Recall that a vector field $f:\mathbb{R}^n\to\mathbb{R}^n$ (or a differential inclusion) is called *homogeneous of degree $\delta\in\mathbb{R}$ with the dilation $d_\kappa:(x_1,\,x_2,\ldots,\,x_n)\mapsto(\kappa^{\rho_1}x_1,\,\kappa^{\rho_2}x_2,\ldots,\,\kappa^{\rho_n}x_n)$*, where $\rho=(\rho_1,\,\rho_2,\ldots,\,\rho_n)$ are some positive numbers (called the *weights*), if for any $\kappa>0$ the identity holds $f(x)=\kappa^{-\delta}d_\kappa^{-1}f(d_\kappa x)$. A scalar function $V:\mathbb{R}^n\to\mathbb{R}$ is called *homogeneous of degree $\delta\in\mathbb{R}$ with the dilation d_κ* if for any $\kappa>0$ the identity holds $V(x)=\kappa^{-\delta}V(d_\kappa x)$.

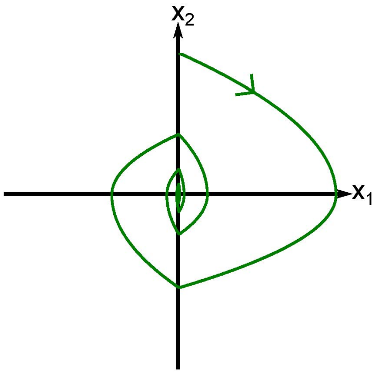

Fig. 1. Integral curve example of the Twisting algorithm

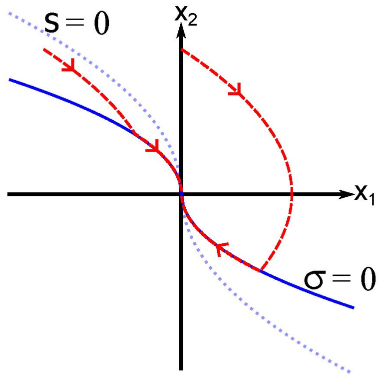

Fig. 2. Phase plane trajectories with the Terminal Algorithm

Another interesting property, shared by many HOSM algorithms, is that they are Piecewise State Affine systems. This property can be very helpful in the analysis and, as we want to propose in this paper, in the construction of Lyapunov functions. To illustrate this fact we will consider some examples.

A. Some Examples

1) Second Order Systems: For the second order system

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = u,$$

consider the generalized 2-sliding homogeneous controller proposed by Levant [10], given by

$$\begin{aligned} u = &-k_1 \operatorname{sign}\left(r_1 x_2 + r_2 \sqrt{|x_1|}\operatorname{sign}(x_1)\right) \\ &-k_2 \operatorname{sign}\left(r_3 x_2 + r_4 \sqrt{|x_1|}\operatorname{sign}(x_1)\right). \end{aligned} \tag{1}$$

This algorithm contains as special cases many of the basic Second Order Sliding Modes (SOSM) Controllers. For example, selecting $r_1 = r_4 = 0$, (1) becomes the Twisting controller [3]

$$u = -k_1 \operatorname{sign}(x_1) - k_2 \operatorname{sign}(x_2). \tag{2}$$

The closed loop system

$$\dot{x}_1 = x_2 \;, \dot{x}_2 = u_i \;, \tag{3}$$

where the constant control values u_i are given by

$$\begin{array}{ll} u_1 = -k_1 - k_2, & x_1 > 0,\, x_2 > 0 \\ u_2 = -k_1 + k_2, & x_1 > 0,\, x_2 < 0 \\ u_3 = k_1 + k_2, & x_1 < 0,\, x_2 < 0 \\ u_4 = k_1 - k_2, & x_1 < 0,\, x_2 > 0 \end{array}$$

is (state) affine (linear with constant imputs) in each of the four regions, in which the state space (x_1, x_2) is divided by the two switching surfaces $\mathcal{S}_1 = \{x_1 = 0\}$ and $\mathcal{S}_2 = \{x_2 = 0\}$. An example of the trajectories of the system with Twisting controller, with gains k_1 and k_2 selected such that asymptotic stability is guaranteed, is shown in Figure 1. Note that the trajectories only cross the switching lines $x_1 = 0$ and $x_2 = 0$ and they never remain over them, i.e. the switching surfaces are not sliding modes.

Now consider the generalized 2-sliding algorithm (1) with $r_1 = r_3$ and $r_2 = r_4$. In this case we obtain the Terminal Algorithm [3]

$$u = -\alpha \operatorname{sign}(\sigma), \quad \sigma = x_2 + \beta\sqrt{|x_1|}\operatorname{sign}(x_1) \tag{4}$$

where $\alpha = k_1 + k_2$ and $\beta = r_2/r_1$. Here the closed loop dynamics (3) is also (state) affine, since u is constant in each of the two regions generated by the switching surface $S = \{\sigma = 0\}$, i.e.

$$u = \begin{cases} u_1 = -\alpha, & \sigma > 0 \\ u_2 = \alpha, & \sigma < 0 \end{cases}.$$

Trajectories of the system with Terminal controller, with stabilizing gains α and β, are shown in Figure 2. Note that for the gains used in the Fig. 2, when the trajectories hit the switching line $\sigma = 0$ they stay on it, i.e. the switching surface is a (first order) sliding mode.

Note also that the generalized 2-sliding algorithm (1) is indeed piecewise state affine for every selection of the parameters r_i, $i = 1, \ldots, 4$.

2) Third Order System: Another example of such a HOSM controller is given by the 3-sliding homogeneous algorithm introduced by [14] for the three dimensional system

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = x_3, \quad \dot{x}_3 = u_\lambda, \tag{5}$$

and given by the event based switching strategy

$$\begin{array}{lll} \texttt{Step 1.} & \texttt{Set}\ \lambda = T & \texttt{until}\ x_2 = x_3 = 0 \\ & u_T = -k_2 \operatorname{sign}(x_2) - k_3 \operatorname{sign}(x_3) & \\ \texttt{Step 2.} & \texttt{Set}\ \lambda = A & \texttt{until}\ x_1 = 0 \\ & u_A = -k_1 \operatorname{sign}(x_1) & \\ \texttt{Step 3.} & \texttt{Go to Step 1} & \end{array}.$$

Figure 3 shows some trajectories for this algorithm. Note that the trajectories only cross the planes $x_1 = 0$, $x_2 = 0$ and $x_3 = 0$ and they never remain on them.

In [14] control u_A is referred as Anosov Unstable (AU) while control u_T is termed Modified Twisting (MTW). Here we call this algorithm AU-MTW. Note again that for the AU-MTW algorithm u is piecewise constant, so that by parts, this is as follows:

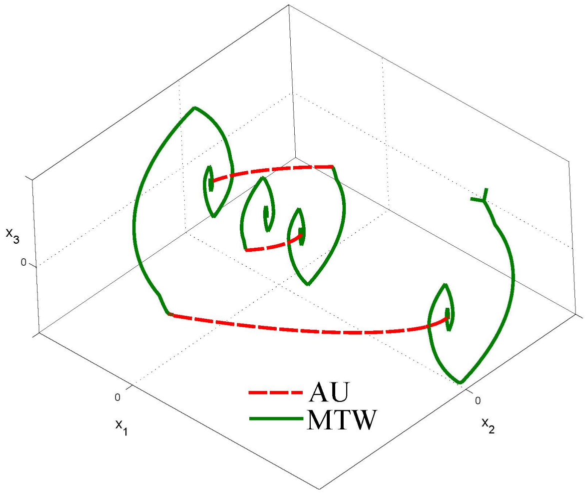

Fig. 3. Integral curve example of the AU-MTW algorithm

When $u = u_T$

$$u = \begin{cases} u_1 = -k_2 - k_3, & x_2 > 0,\, x_3 > 0 \\ u_2 = -k_2 + k_3, & x_2 > 0,\, x_3 < 0 \\ u_3 = k_2 + k_3, & x_2 < 0,\, x_3 < 0 \\ u_4 = k_2 - k_3, & x_2 < 0,\, x_3 > 0 \end{cases}$$

and when $u = u_A$

$$u = \begin{cases} u_5 = -k_1, & x_1 > 0 \\ u_6 = k_1, & x_1 < 0 \end{cases}.$$

B. A More General Class of Systems

The given HOSM controllers are examples of a more general class of algorithms (see e.g. [3], [9]) that can be applied to an n-th order system

$$\begin{array}{rcl} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & x_3 \\ \vdots & \vdots & \vdots \\ \dot{x}_{n-1} & = & x_n \\ \dot{x}_n & = & f(t,x) + u \end{array} \tag{6}$$

where the state $x \in \mathbb{R}^n$, $f(t,x) \in \mathbb{R}$ is an uncertain function such that $|f(t,x)| \leq F$ for any $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$, for a known constant $F \in \mathbb{R}$, and $u \in \mathbb{R}$ is the control input, that is, for example, piecewise constant, leading to a closed loop system that is state affine, and possibly homogeneous. Note that the solutions of system (6) when a HOSM controller is applied are to be understood in the sense of Filippov [7], and so they are absolutely continuous functions of time.

This dynamics (6) can be written as (we set by now here $f(t,x) = 0$) a set of m state affine systems given by

$$\dot{x} = Ax + Bu_i, \quad \forall x \in \mathcal{S}_i \tag{7}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & \vdots \\ \vdots & & \ddots & & 0 \\ 0 & & & 0 & 1 \\ 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

and u_i, $i \in \{1, 2, \ldots, m\}$, is constant on each set $\mathcal{S}_i$. Their boundaries correspond to the switching surfaces.

Let $\phi_i(\tau; t, x)$ denote the solution of (7) whose initial condition is the point $x \in \mathcal{S}_i$ in the time t, this means that $\phi_i(t; t, x) = x$.

Now, on the interior of each set $\mathcal{S}_j$, $j \in \{1, 2, \ldots, m\}$, the system (7) has a solution $\varphi_j(t; 0, x)$, with initial condition x at the instant of time $t = 0$. This trajectory is given by

$$\varphi_j(t; 0, x) = e^{At}x + \int_0^t e^{A(t-\tau)} d\tau B u_i \,.$$

Since

$$e^{At} = \begin{bmatrix} 1 & t & \frac{t^2}{2!} & \cdots & \frac{t^{n-1}}{(n-1)!} \\ 0 & 1 & t & & \vdots \\ \vdots & & \ddots & & \frac{t^2}{2!} \\ 0 & & & 1 & t \\ 0 & \cdots & \cdots & \cdots & 1 \end{bmatrix}, \; e^{At}B = \begin{bmatrix} \frac{t^{n-1}}{(n-1)!} \\ \frac{t^{n-2}}{(n-2)!} \\ \vdots \\ t \\ 1 \end{bmatrix},$$

the expression for $\varphi_j(t; 0, x)$ is polynomial in t. If there are no sliding surfaces, then the whole trajectory of the system $\phi_i(\tau; t, x)$ is a sequence of such partial trajectories. When there are sliding surfaces, it is necessary to calculate their particular behavior. Note that due to the homogeneity of the system, the solution function $\phi_i(\tau; t, x)$ will be also homogeneous (see for example Proposition 5.8 of [4]).

C. Construction of Lyapunov Functions

The construction of Lyapunov functions for this kind of HOSM algorithms cannot proceed as for some classes of hybrid or piecewise linear systems, where some of the subsystems has a stable equilibrium point (at the origin), and a continuous or discontinuous Lyapunov function is designed by combining these functions. In the present case the origin is not even an equilibrium point for any of the subsystems. Moreover, many HOSM algorithms (as e.g. the Twisting algorithm) present the Zenon's phenomenon.

It is well known, for example from the constructive proofs of the Converse of Lyapunov's Theorem (see for example Chapter 4 of [6]), that integrating e.g. the norm of the state along the whole trajectory of the system leads to a Lyapunov function for the system. More specifically, the function

$$V(x) = \begin{cases} \int_t^{t+\delta} W\left(\phi_1(\tau; t, x)\right) \mathrm{d}\tau, & x \in \mathcal{S}_1 \\ \vdots & \vdots \\ \int_t^{t+\delta} W\left(\phi_m(\tau; t, x)\right) \mathrm{d}\tau, & x \in \mathcal{S}_m \end{cases} \tag{8}$$

where $W(x_1, \ldots, x_n)$ is a positive definite function, is a Lyapunov function for system (7) for some δ. Note that in (8) the intersection of the sets $\mathcal{S}_i$ must be empty and their union equal to $\mathbb{R}^n$. Some characteristics of this method are:

- Since a converging HOSM algorithm has negative homogeneity degree, its convergence occurs in finite time [9], [10], so that if $\delta \to \infty$ then the integral does converge for any $W(x)$.
- If $W(x)$ is continuous then, due to the continuity of solutions of a differential inclusion [7] with respect to the

initial condition, the function $V(x)$ is also continuous when $\delta \to \infty$. It is also differentiable for every x not on the switching surfaces $\mathcal{S}_i$.

- If $W(x)$ is homogeneous then $V(x)$ will be also homogeneous. This is due to the homogeneity of the system.
- The derivative of the function $V(x)$ along the trajectories of system (7) is

$$\dot{V} = -W(x)\,.$$

it is therefore clear that $V(x)$ is positive definite and that $\dot{V}$ is negative definite.

- If the integral does not converge for any positive definite function $W(x)$, then the algorithm is unstable. This is, one obtains necessary and sufficient conditions for stability.
- The homogeneity degree of $W(x)$ is lower than the homogeneity degree of $V(x)$ by a fixed number. Therefore, the higher the homogeneity degree of $W(x)$ is selected, the higher (and smoother) the resulting Lyapunov function $V(x)$ will be.
- The method can be extended to the case with perturbations, i.e. $f(t,x) \neq 0$.

As described before, it is possible to have an explicit expression for the trajectories of system (7), so that in this case the construction of $V(x)$ in (8) can be feasible. We will show by means of the examples presented in the next sections, that this is indeed the case.

III. Lyapunov functions for Twisting algorithm

Consider (6) with $f(t,x) = 0$ and u as in (2), that is

$$\begin{array}{rcl} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & -k_1 \operatorname{sign}(x_1) - k_2 \operatorname{sign}(x_2). \end{array} \quad (9)$$

Note that this system is homogeneous of degree $\delta = -1$ with weights $[\rho_1, \rho_2] = [2, 1]$.

A. Homogeneous Lyapunov function of degree 1

Applying the method described in Section II by setting the homogeneus function of degree 0 (for simplicity 0-homogeneous)

$$W(x) = 1,$$

it results that each integral in (8) equals δ. Selecting δ as the convergence time to the origin we get the following Lyapunov function for (9)

$$V(x) = \begin{cases} \eta_1|x_2| + \eta_2\Gamma_1, & x_1x_2 > 0 \\ \eta_3|x_2| + \eta_4\Gamma_2, & x_1x_2 \leq 0 \end{cases} \quad (10)$$

where

$$\Gamma_1 = \sqrt{x_2^2 + 2(k_1+k_2)|x_1|}, \quad \Gamma_2 = \sqrt{x_2^2 + 2(k_1-k_2)|x_1|}.$$

$$\eta_1 = \frac{1}{k_1+k_2}, \; \eta_2 = \frac{2k_1\eta_1\left(\sqrt{k_1-k_2}\right)^{-1}}{\sqrt{k_1+k_2} - \sqrt{k_1-k_2}},$$

$$\eta_3 = -\frac{1}{k_1-k_2}, \; \eta_4 = \frac{-2k_1\eta_3\left(\sqrt{k_1+k_2}\right)^{-1}}{\sqrt{k_1+k_2} - \sqrt{k_1-k_2}}.$$

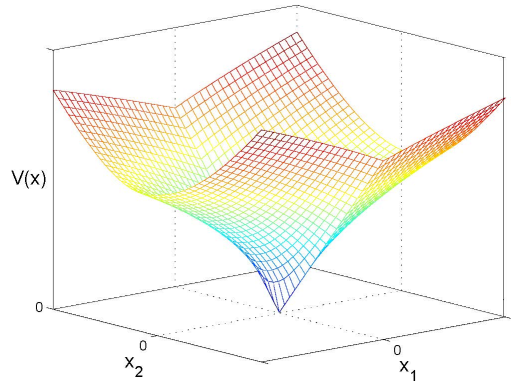

Fig. 4. 1-homogeneus LF for Twisting algorithm

Since the integral on the right hand of (8) converges only for $k_1 > k_2 > 0$, then these conditions are *necesary and sufficient* for the finite time stability of the point $x = 0$ (without perturbation). A graph of the 1-homogeneus Lyapunov Function (LF) $V(x)$ is shown in Figure 4 for $k_1 = 2$ and $k_2 = 1$.

Function $V(x)$ in (10) is homogeneous of degree $\delta_V = 1$ with weights $[\rho_1, \rho_2] = [2, 1]$. Moreover, it is continuous everywhere but not differentiable on the switching lines $\mathcal{S}_1 = \{x_1 = 0\}$ and $\mathcal{S}_2 = \{x_2 = 0\}$. Since there are no sliding trajectories on these switching lines (see also Fig. 1) it follows that the function $V(x)$ evaluated along a trajectory of the system $\phi(t;0;x)$, i.e. $V(\phi(t;0;x))$ is differentiable almost everywhere (except at the points where one of the switching surfaces is crossed), and, since (10) was obtained selecting $W(x) = 1$, its time derivative is almost everywhere

$$\dot{V} = -1. \quad (11)$$

Integrating both sides of (11) from the initial time $t_0 = 0$ to the time $t = T > 0$, it is obtained that

$$V(x(T)) - V(x(0)) = -T. \quad (12)$$

It is clear from (10) that $V(x(0)) < \infty$. Since system (9) is homogeneous of negative degree, it converges in finite time. If we consider that the convergence time is T, i.e. $V(x(T)) = 0$, then from (12) we have that

$$V(x(0)) = T < \infty$$

which shows that the reaching time to $x = 0$ is exactly $V(x(0))$.

B. Homogeneous Lyapunov function of degree 3

If we now choose the (continuous) 2-homogeneus function

$$W(x) = |x_1| + x_2^2,$$

then, applying the method, the Lyapunov Function obtained for (9) is

$$V(x) = \begin{cases} \lambda_1|x_2|^3 + \lambda_2x_1x_2 + \lambda_3\Gamma_1^3, & x_1x_2 > 0 \\ \lambda_4|x_2|^3 + \lambda_5x_1x_2 + \lambda_6\Gamma_2^3, & x_1x_2 \leq 0 \end{cases} \quad (13)$$

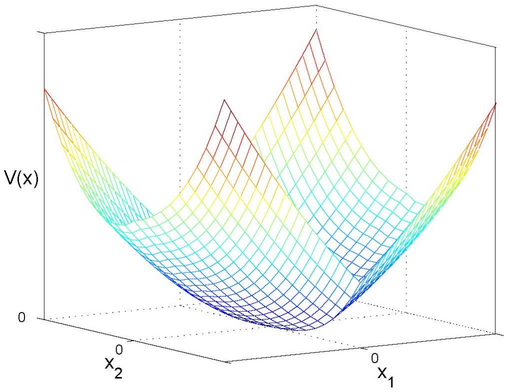

Fig. 5. 3-homogeneus LF for Twisting algorithm

with Γ_1, Γ_2 defined as before and

$$\lambda_1 = \frac{k_1+k_2+1}{3(k_1+k_2)^2}, \quad \lambda_2 = \frac{1}{k_1+k_2},$$

$$\lambda_3 = \frac{2\lambda_2^2\left[k_1^2(k_1+1) - k_2^2(k_1-1)\right]}{3\sqrt{k_1-k_2}\left[(k_1+k_2)^{\frac{3}{2}} - (k_1-k_2)^{\frac{3}{2}}\right]},$$

$$\lambda_4 = -\frac{k_1-k_2+1}{3(k_1-k_2)^2}, \quad \lambda_5 = \frac{1}{k_1-k_2},$$

$$\lambda_6 = \lambda_1 + \lambda_3 - \lambda_4.$$

A graph of the 3-homogeneus LF $V(x)$ is shown in Figure 5 for $k_1 = 2$ and $k_2 = 1$.

Function (13) is continuous everywhere, and it is differentiable almost everywhere, except on the switching surface $\mathcal{S}_1$. However it is locally Lipschitz continuous everywhere. Therefore $V(\phi(t;0;x))$, being the composition of two Lipschitz continuous functions, is also a Lipschitz continuous function of time [5, p. 391], having a derivative almost everywhere, which is given by

$$\dot{V} = -W(x) = -\left(|x_1| + x_2^2\right). \tag{14}$$

From the differential equation (14) it is possible to derive a Differential Inequality for V of the form

$$\dot{V} \le -\gamma V^{\frac{2}{3}}.$$

From this inequality, and using the comparison lemma, an upper bound for the convergence time T to the origin can be calculated as follows

$$T \le 6(k_1+k_2)\lambda_3^{\frac{2}{3}} V^{\frac{1}{3}}(x(0)).$$

C. Twisting Algorithm with perturbations

In order to deal with the disturbed case, consider (6) with $f(t,x) \neq 0$ and u as in (2), this is

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = f(t,x) + u. \tag{15}$$

It is possible to derive *sufficient* conditions for (robust) stability of the origin $x = 0$ using the same Lyapunov function $V(x)$ of the unperturbed case (10). Taking its time derivative along the trajectories of the perturbed system (15) we find that (10) is a Lyapunov Function for(15) provided that k_1 and k_2 are chosen such that they satisfy the next condition

$$\frac{(k_1-k_2)\left(\sqrt{k_1+k_2} - \sqrt{k_1-k_2}\right)}{3k_1+k_2-\sqrt{k_1^2-k_2^2}} > F,$$

where F is a uniform bound of the perturbation, i.e. for all (t,x) the condition $|f(t,x)| \le F$ is satisfied. The (finite) convergence time $T(x(0))$ can be estimated as

$$T \le \tfrac{1}{c} V(x(0)),$$

where

$$c = 1 - \frac{3k_1+k_2-\sqrt{k_1^2-k_2^2}}{(k_1-k_2)\left(\sqrt{k_1+k_2} - \sqrt{k_1-k_2}\right)} F.$$

IV. Lyapunov Function for a Third Order Algorithm

To illustrate the fact that the proposed construction method is not restricted to systems on the plane ($n = 2$), let us consider (6) with $n = 3$, $f(t,x) = 0$ and u given by the AU-MTW algorithm described in Section II. Note that (5) is a hybrid system with a continous state $x = (x_1, x_2, x_3)^T$ and a discrete state λ whose discrete values are T and A such as is stated by the AU-MTW algorithm. Selecting

$$W(x) = 1,$$

and applying the method, we get the following LF considering the initial control as u_T but also including the pieces of the trajectories with the control u_A

$$V(x) = \begin{cases} \rho_1|x_3| + \rho_2\Theta_1 + \rho_3\left|\Sigma_1\right|^{1/3}, & x \in \mathcal{R}_1 \\ \rho_1|x_3| + \rho_2\Theta_1 + \rho_3\left|\Sigma_2\right|^{1/3}, & x \in \mathcal{R}_2 \\ -\rho_4|x_3| + \rho_5\Theta_2 + \rho_3\left|\Sigma_3\right|^{1/3}, & x \in \mathcal{R}_3 \\ -\rho_4|x_3| + \rho_5\Theta_2 + \rho_3\left|\Sigma_4\right|^{1/3}, & x \in \mathcal{R}_4. \end{cases} \tag{16}$$

where

$$\begin{aligned} \Sigma_1 &= x_1 + \rho_1 x_2 x_3 + (\rho_1^2/3)x_3^3 + (R\rho_1^2/3)\Theta_1^3, \\ \Sigma_2 &= x_1 - \rho_1 x_2 x_3 + (\rho_1^2/3)x_3^3 - (R\rho_1^2/3)\Theta_1^3, \\ \Sigma_3 &= x_1 - \rho_4 x_2 x_3 + (\rho_4^2/3)x_3^3 - (rR\rho_4^2/3)\Theta_2^3, \\ \Sigma_4 &= x_1 + \rho_4 x_2 x_3 + (\rho_4^2/3)x_3^3 + (rR\rho_4^2/3)\Theta_2^3, \end{aligned}$$

$$\mathcal{R}_1 = \{x_2 \ge 0, x_3 \ge 0\}, \ \mathcal{R}_2 = \{x_2 \le 0, x_3 \le 0\},$$
$$\mathcal{R}_3 = \{x_2 < 0, x_3 > 0\}, \ \mathcal{R}_4 = \{x_2 > 0, x_3 < 0\},$$

$$\Theta_1 = \sqrt{x_3^2 + 2(k_2+k_3)|x_2|}, \ \Theta_2 = \sqrt{x_3^2 + 2(k_2-k_3)|x_2|},$$

$$\rho_1 = \frac{1}{k_2+k_3}, \quad \rho_2 = \frac{2k_2\rho_1(k_2-k_3)^{-\frac{1}{2}}}{\sqrt{k_2+k_3} - \sqrt{k_2-k_3}}$$

$$\rho_3 = \frac{1 + \rho_1 k_1 + \rho_2\sqrt{k_1^2 + k_1(k_2+k_3)}}{(k_1/6)^{1/3}(1 - C^{1/3})},$$

$$\rho_4 = \frac{1}{k_2-k_3}, \ \rho_5 = \frac{2k_2\rho_4(k_2+k_3)^{-\frac{1}{2}}}{\sqrt{k_2+k_3} - \sqrt{k_2-k_3}},$$

$$r = \sqrt{\frac{k_2-k_3}{k_2+k_3}}, \ R = \frac{1 - r^3 - r^4 + r^7}{r - r^7},$$

$$C = \frac{2R\sqrt{k_1(k_1+k_2+k_3)^3} + k_1(2k_1 + 3(k_2+k_3))}{(k_2+k_3)^2}.$$

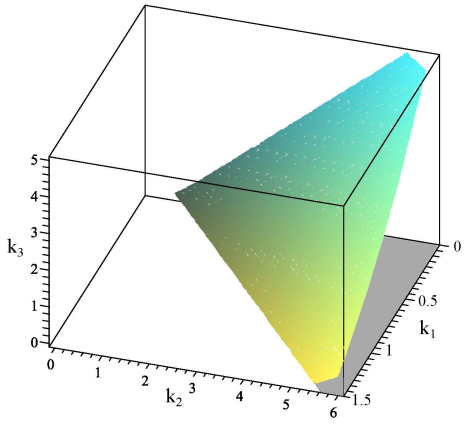

Fig. 6. Surface $C = 1$ in the gain space for the AU-MTW Algorithm. The Stability region is on the right of the surface.

Since the integral on the right hand of (8) converges only when $C < 1$, then this condition is *necesary and sufficient* for the finite time stability of the point $x = 0$ (without perturbation). In Figure 6 the surface $C = 1$ is presented in the gain space (k_1, k_2, k_3). Only for the values on the right of this surface finite time stability is assured.

Function (16) is continuous everywhere but not differentiable on the planes $x_1 = 0$, $x_2 = 0$ and $x_3 = 0$. Note that the trajectories of the system only cross these planes and they never remain there (see also Fig. 3). Therefore $V(\phi(t; 0; x))$ is differentiable (as a function of time) almost everywhere (except on the time instants where the trajectory crosses one of the planes $x_i = 0$, $i = 1, 2, 3$). Its derivative along the trajectories of the system with the control u_T is almost everywhere $\dot{V} = -1$.

Now consider the surface

$$\begin{aligned} \mathcal{R}_A &= \{(x_1 \le 0,\ x_2, x_3 \ge 0,\ x_3^2 = 2k_1 x_2) \\ &\quad \text{or } (x_1 \ge 0,\ x_2, x_3 \le 0,\ x_3^2 = -2k_1 x_2)\}. \end{aligned}$$

and note that in order to take the derivative along the trajectories of the system when control u_A is being applied it is needed another expresion of $V(x)$. Considering the fact that trayectories with u_A only take values in the surface $\mathcal{R}_A$ and applying the construction method we get

$$V_A(x) = \rho_3 \left(|x_1| + \frac{|x_3|^3}{6k_1^2} \right)^{\frac{1}{3}} - \frac{1}{k_1}|x_3|, \quad x \in \mathcal{R}_A \qquad (17)$$

whose time derivative is given again by $\dot{V} = -1$. Thus the reaching time to the origin T can be calculated exactly as follows

$$T = V(x(0)).$$

We can also deal with the perturbed case in the same manner as it was done with the 1-homogeneus LF for Twisting algorithm.

V. Conclusions

A method to construct (strict) Lyapunov Functions for a class of Higher Order Sliding Modes (HOSM) algorithms, that are homogeneous and piecewise state affine has been presented. Its construction is based in the knowledge of an expression for the solutions of the system, that leads to a Lyapunov function, in the same spirit as in the converse Lyapunov's theorems. Although this method can be applied to any system in principle, it is only feasible in some cases, as for Linear Time Invariant systems. We show by means of two examples that this is also true for some classes of HOSM algorithms, and we have also been able to construct Lyapunov functions using this idea for some other algorithms. This will be reported somewhere else. We also hope that this method will be able to provide with Lyapunov functions for some important HOSM algorithms, what is an important complement to this research area.

VI. Acnowledgments

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